Newtonian approach and Friedmannien equations

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Friedmannien equations

The metric interval is:

FLRW metric is :

$$ds^2 = c^2 dt^2 - a^2(t)\left(dr^2 + f^2(r)\left\{d\theta^2 + \sin^2\theta d\varphi^2\right\}\right)$$

here $f(r)$ is a function which determines global geometrical property of 3D space.

One can put it in the Einstein equations and obtains equation Friedmannien equations which describe the evolution of our Universe.

and three Friedmannien equations are:

$$\frac{1}{2}\dot{a}^2 = \frac{4\pi}{3}Ga^2\rho - \frac{kc^2}{2}$$

$$\ddot{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)a$$

$$\frac{d\rho}{dt} = -3H(\rho + \frac{p}{c^2})$$

and one more is the equation of state

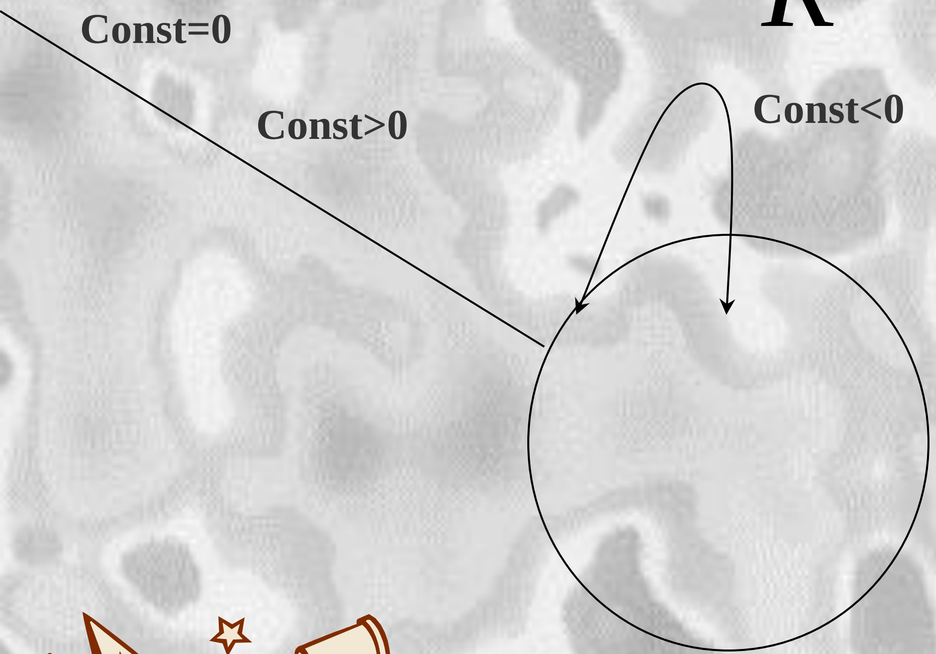
$$p = q\rho c^2$$

Motion of test particle

- Let consider the motion of a test particle launched from the Earth. If the period of acceleration is short enough, one can consider the motion of the particle as motion in the Earth's gravitational field only.

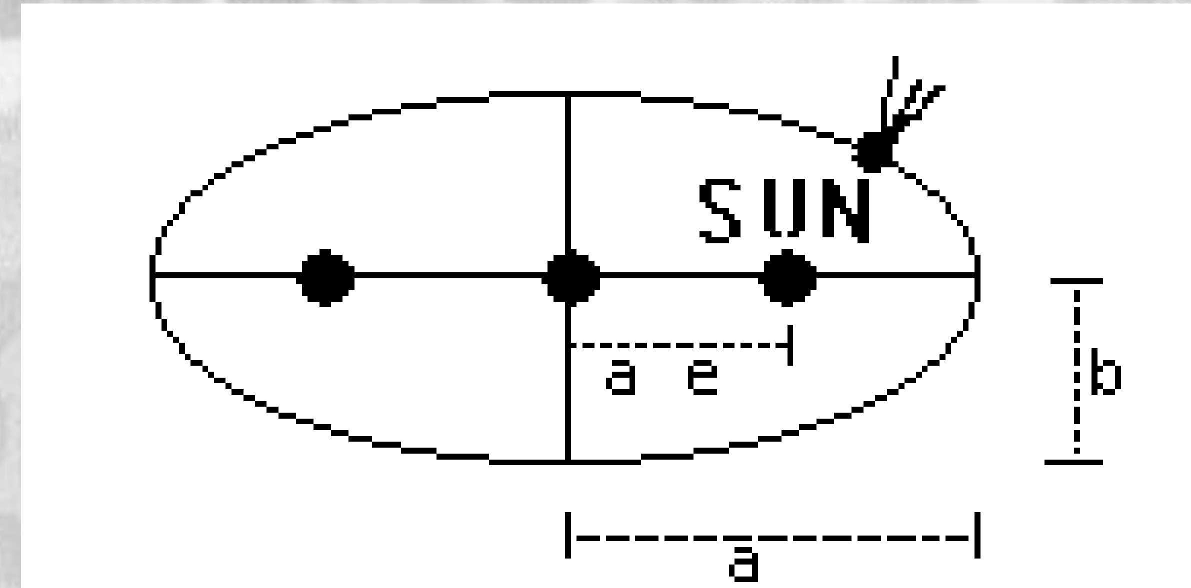
The equation of energy conservation provides us with equation like

$$v^2 - \frac{2GM_{\oplus}}{R} = const$$



1. Const >0 is infinite motion
2. Const=0 is infinite motion with escaping velocity
3. Const<0 is finite motion

A planet moves around the Sun in elliptic trajectory. The Sun is in a focus of the ellipse.



Here is attraction body (the Sun) and a planet which is moving the gravitational field of the Sun (in fact, in sum of the two fields: the Sun and the planet).

Our Universe is filled with matter media. The matter in the Universe is distributed homogeneous and isotropic. Homogeneity is independence of the main physical matter characteristics of position in space, while isotropy is a independence of that in different directions. So, the motion of the matter is not the same as motion of a planet around the Sun.

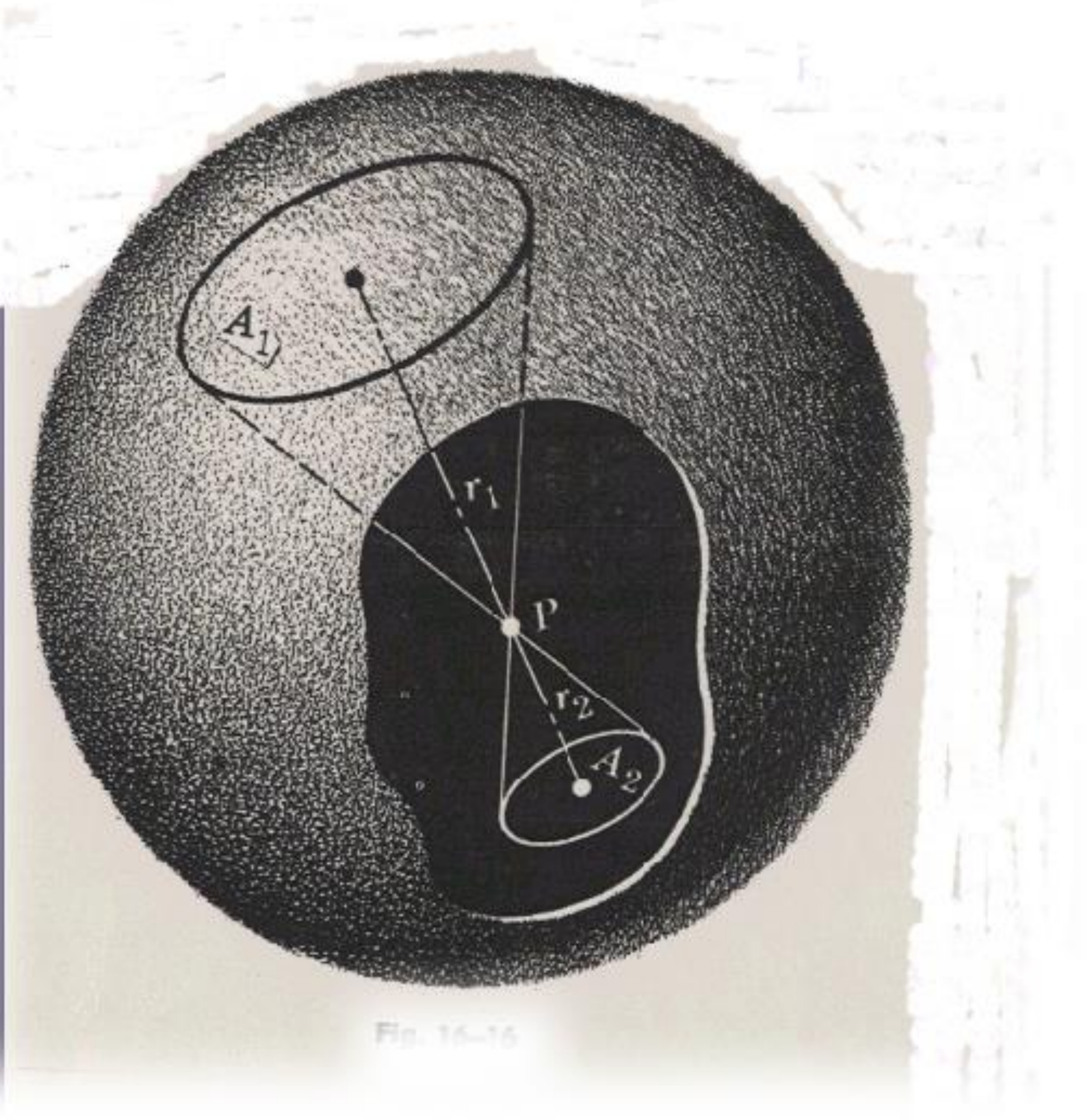
$$\Delta\varphi = 4\pi G\rho(t)$$

How one can solve it in the case of homogeneous density?

The motion of the Universe (filled homogeneously by matter) is similar, but not exactly the same as keplerian motion of celestrial bodies in our solar system.

We have to consider the motion of matter media in gravitational field of the media.

The solution in ordinary integral diverges.



$$\vec{F}_1 = -\frac{GM_1}{r_1^{\ 2}}\vec{n}_1$$

$$\vec{F}_2 = -\frac{GM_2}{r_2^{\ 2}}\vec{n}_2$$

$$\vec{n}_1 = -\vec{n}_2$$

$$M_1 = \rho A_1 \Delta r \qquad M_2 = \rho A_2 \Delta r$$

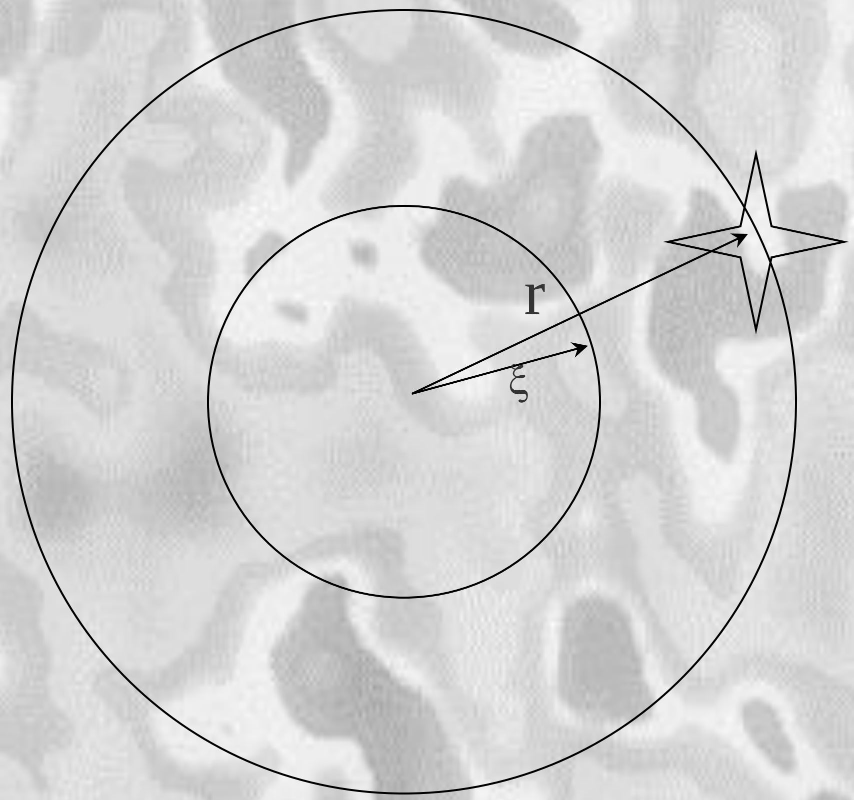
$$A_1 = r_1^2 d\Omega \qquad A_2 = r_2^2 d\Omega$$

$$\vec{F}_1 = -\rho d\Omega \Delta r \vec{n}_1 \quad \vec{F}_2 = -\rho d\Omega \Delta r \vec{n}_2$$

$$\vec{F}_1 + \vec{F}_2 = -\rho d\Omega \Delta r (\vec{n}_1 + \vec{n}_2)$$

$$= \vec{0}$$

$r(t)=a(t)\xi$ here r is distance to test particle and ξ is lagrangian coordinate of this particle, a(t) is called scale factor.



Equation of energy conservation

$$\frac{1}{2}\left(\frac{dr(t)}{dt}\right)^2 - \frac{GM}{r(t)} = const = \mathrm{K}$$

$$M = \rho V = \frac{4\pi}{3} r^3(t)\rho \qquad M = \rho V = \frac{4\pi}{3} a^3(t)\rho\xi^3$$

$$\frac{1}{2}\dot{a}^2 - \frac{4\pi}{3} G a^2 \rho = \frac{\mathrm{K}}{\xi^2}$$

First Friedmannien equation

Thermodynamical Relations (equations)

Here we will use equations of the special relativity (SR) and thermodynamical equations. The equation of SR is

$$E = Mc^2$$

energy- mass equation

$$dE + pdV = 0$$

entropy conservation

$$E = Mc^2 = \rho V c^2$$

differential of both sides of the equation is

$$dE = V d\rho c^2 + \rho dV c^2$$

using the entropy conservation equation and substitute it
one can get equation for density evolution or
third Friedmannien equation

$$p dV + V d\rho c^2 + \rho dV c^2 = 0$$ or

$$\frac{d\rho}{dt} = -(\rho + \frac{p}{c^2})\frac{dV}{V dt}$$

if one remind the equation for volume

$$V = \frac{4\pi}{3} a^3(t)\xi^3$$

volume differential over volume is

$$\frac{dV}{Vdt} = 3\frac{1}{a}\frac{da}{dt}$$

Let introduce the definition $\frac{1}{a}\frac{da}{dt} = H$ and now

$$\frac{d\rho}{dt} = -3H(\rho + \frac{p}{c^2})$$

How one can obtain the equation of a rocket motion?

$$v^2 - \frac{2GM_\oplus}{R} = const$$

One can apply the differentiation with respect to time to both sides of this equation

$$\frac{d}{dt}\left(\frac{1}{2}v^2\right) - \frac{d}{dt}\left(\frac{GM_\oplus}{r}\right) = 0$$

Differentiation provides us with equation of motion

The first term reads

$$\frac{d}{dt}\left(\frac{1}{2}v^2\right) = \frac{1}{2}2v\dot{v} = v\frac{d^2r}{dt^2}$$

and the second is

$$\frac{d}{dt}\left(\frac{GM_{\oplus}}{r}\right) = -\frac{GM_{\oplus}}{r^2}v$$

here $M_{\oplus} = const$ is constant with respect to time velocity is cancelled in both sides, so one remains with the equation of motion

$$\ddot{r} = -\frac{G M_{\oplus}}{r^2}$$

minus sign designate attraction

The same procedure can be applid to obtain cosmological equation of motion

$$\frac{d}{dt}\left(\frac{1}{2}\dot{r}^2\right) - \frac{d}{dt}\left(\frac{GM}{r}\right) = 0$$

$$\frac{d}{dt}\left(\frac{1}{2}\dot{r}^2\right) = \dot{r}\ddot{r}$$

The second term is a bit difficult. The mass is no more constant. The M is variable, it is function of time.

$$\frac{d}{dt}\left(\frac{GM}{r}\right) = -\frac{GM}{r^2}\dot{r} + \frac{G\dot{M}}{r}$$

To calculate derivative of mass with respect to time we have remember the thermodynamical and SR equations.

$$E = Mc^2 \qquad dE + pdV = 0$$

from these follow

$$dM = -\frac{p}{c^2}dV$$

from that follows

$$\ddot{r}r + \frac{GM}{r^2}\dot{r} + \frac{G}{r}\frac{p}{c^2}\frac{dV}{dt} = 0$$

the derivative of volume with respect to time one can do as follows

$$\frac{dV}{dt} = 4\pi r^2 \dot{r}$$

and after this procedure one can cancel velocity $\dot{r}$

and obtain the following equation

$$\ddot{r} = -\frac{GM}{r^2} - 4\pi G \frac{p}{c^2} r$$

the acceleration of a test particle on the surface of a cut sphere is then

$$\ddot{r} = -\frac{4\pi G}{c^2}(\rho + \frac{3p}{c^2})r$$

in terms of density and pressure

One can substitute eulerian coordinate r(t) by lagrangian coordinate ξ according to equation r(t) = a(t) ξ and obtain the second Friedmannien equation:

$$\ddot{a} = -\frac{4G}{3}\left(\rho + \frac{3p}{c^2}\right)a$$

In fact, eulerian coordinate is changed during the Universe evolution and lagrangian coordinate does not changed. So, we can conclude that spatial coordinate ξ forms a comoving coordinate system in the sense that typical galaxy has constant lagrangian coordinate.

The dust dominated Universe p=0

$$\ddot{r} = -\frac{4\pi G}{c^2}\rho r$$

Negative sign = attraction

The radiation dominated Universe p=ε/3

$$\ddot{r} = -\frac{8\pi G}{c^2}\rho r$$

Negative sign = attraction. The mixture of these types of matter produces attraction.

Let us consider negative relativistic pressure

$$p = -\rho c^2$$

in this case

$$\rho + \frac{3p}{c^2} = -2\rho$$

and gravitational attraction is exchanged by gravitational repulsion

$$\ddot{r} = +\frac{8\pi G}{c^2}\rho r$$

The analysis of solution of Friedmannien equations

The metric interval is:

$$ds^2 = c^2 dt^2 - a^2(t)\left(dr^2 + f^2(r)\left\{d\theta^2 + \sin^2\theta d\varphi^2\right\}\right)$$

and three Friedmannien equations are:

$$\frac{1}{2}\dot{a}^2 = \frac{4\pi}{3}Ga^2\rho - \frac{kc^2}{2}$$

$$\ddot{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)a$$

$$\frac{d\rho}{dt} = -3H(\rho + \frac{p}{c^2})$$

$$p = q\rho c^2$$

The dust dominated Universe

p=0 and our equation become

$$\frac{1}{2}\dot{a}^2 = \frac{4\pi}{3}Ga^2\rho - \frac{kc^2}{2}$$

$$\ddot{a} = -\frac{4\pi G}{3}\rho a$$

$$\frac{d\rho}{dt} = -3H\rho$$

The third equation can be rewritten as follows

$$\frac{d}{dt}\left(\rho a^{3}\right) = 0$$

and the solution is

$$\rho(t) = \rho_0 \frac{a_0^3}{a^3(t)}$$

The physical sense of the equation is

$$\rho(t) = mn(t) \qquad m = const$$

$$n(t) = \frac{N}{V(t)} \qquad V(t) = \frac{4\pi}{3}\xi^3 a^3(t)$$

or

$$n(t)a^3(t) = const$$

$$\frac{kc^2}{2} = \frac{4\pi}{3}Ga^2\rho - \frac{1}{2}\dot{a}^2$$

$$\frac{kc^2}{2} = \frac{4\pi}{3}Ga_0^2\left(\rho_0 - \frac{3}{8\pi G}H^2\right) \qquad H = \frac{\dot{a}_0}{a_0}$$

$$\rho_{crit} = \frac{3}{8\pi G}H^2$$

$$\frac{kc^2}{2} = \frac{4\pi}{3} G a_0^2 \left(\rho_0 - \rho_{crit} \right)$$

If k>0 then $\rho_0 > \rho_{crit}$

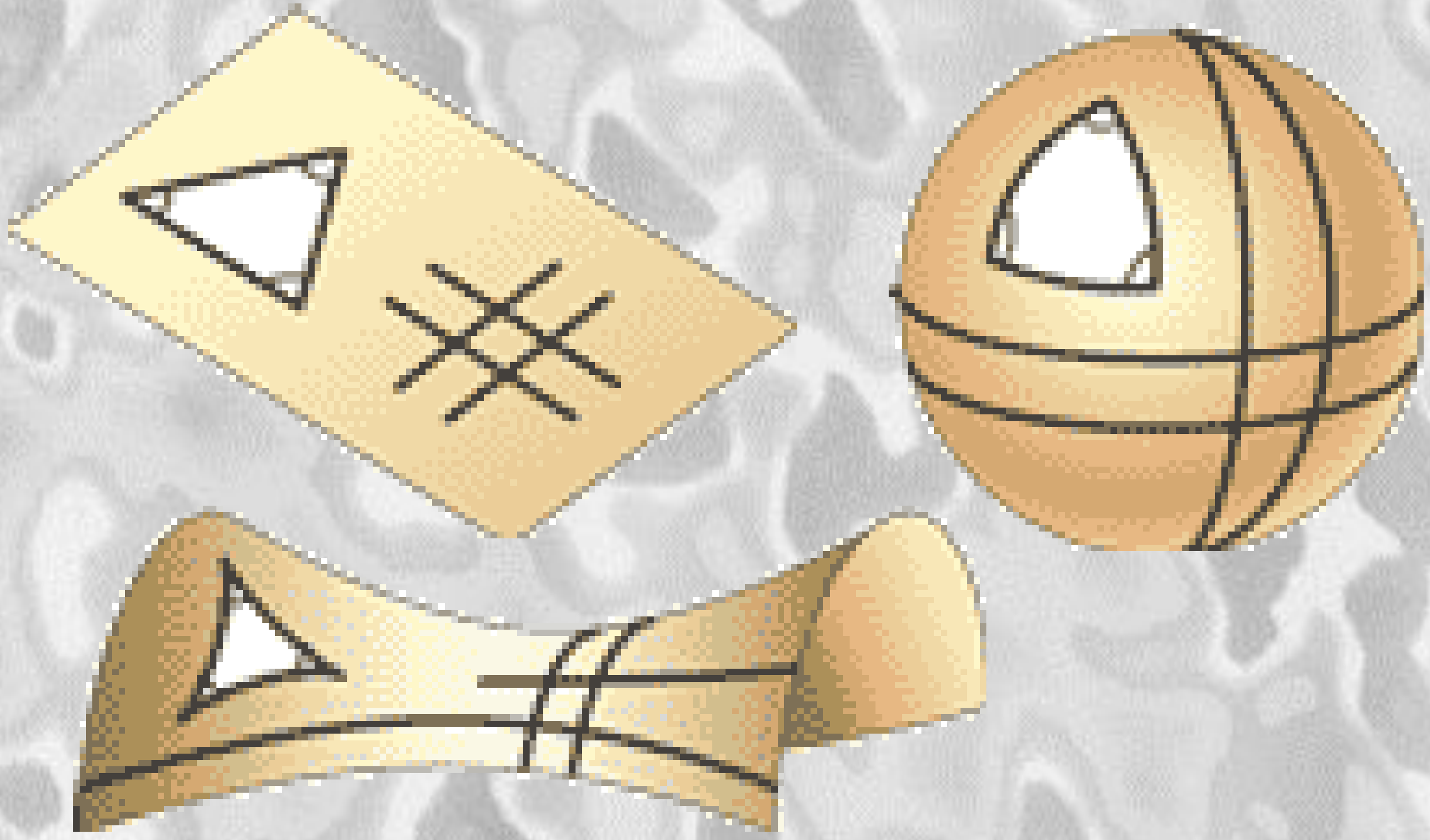
and our Universe is closed and finite in volume

If k=0 then $\rho_0 = \rho_{crit}$

and our Universe is flat and infinite in volume

If k<0 then $\rho_0 < \rho_{crit}$

and our Universe is open and infinite in volume



If k=0 the equation becomes

$$\frac{1}{2}\left(\frac{da}{dt}\right)^2 = \frac{GM}{a}$$

where $$M = \frac{4\pi}{3}\rho_0 a_0^3$$

and solution is

$$a(t) = \left(\frac{3}{2}\right)^{\left(2/3\right)} R^{1/3}{}_g\left(t - t_{in}\right)^{2/3}$$

and

$$a(t) = a_0\left(\frac{t}{t_0}\right)^{2/3}$$

The Ω parameter

One can introduce the Ω parameter which is more convenient in many cases

$$\Omega = \frac{\rho}{\rho_{crit}}$$

$\Omega > 1$ the closed Universe

$\Omega = 1$ the flat Universe

$\Omega < 1$ the open Universe

$$\frac{kc^2}{2} = \frac{1}{2} H_0^2 a_0^2 (\Omega_0 - 1)$$

Ω_0 is for the present density parameter

If we have a Universe filled with different types of matter Ω parameter is sum of several contribution.

$$\Omega_0 = \Omega_{0m} + \Omega_{0\Lambda} + \Omega_{0r} + \ldots$$

The radiation dominated Universe

$p = \rho c^2/3$ and k=0 Friedmannien equations become

$$\frac{1}{2}\dot{a}^2 = \frac{4\pi}{3} G a^2 \rho$$

$$\ddot{a} = -\frac{4\pi G}{3} \rho a$$

$$\frac{d\rho}{dt} = -4H\rho$$

The third equation can be rewritten as follows

$$\frac{d}{dt}\left(\rho a^4\right) = 0$$

and the solution is $\rho(t) = \rho_0 \dfrac{a_0^4}{a^4(t)}$

The physical sense of the equation is

$\rho(t) = mn(t)$ $m \propto 1/a(t)$ and

$$n(t) \propto 1/a^3(t)$$

photon’s mass red shifted during expansion.
Therefore, this is valid for relativistic particles

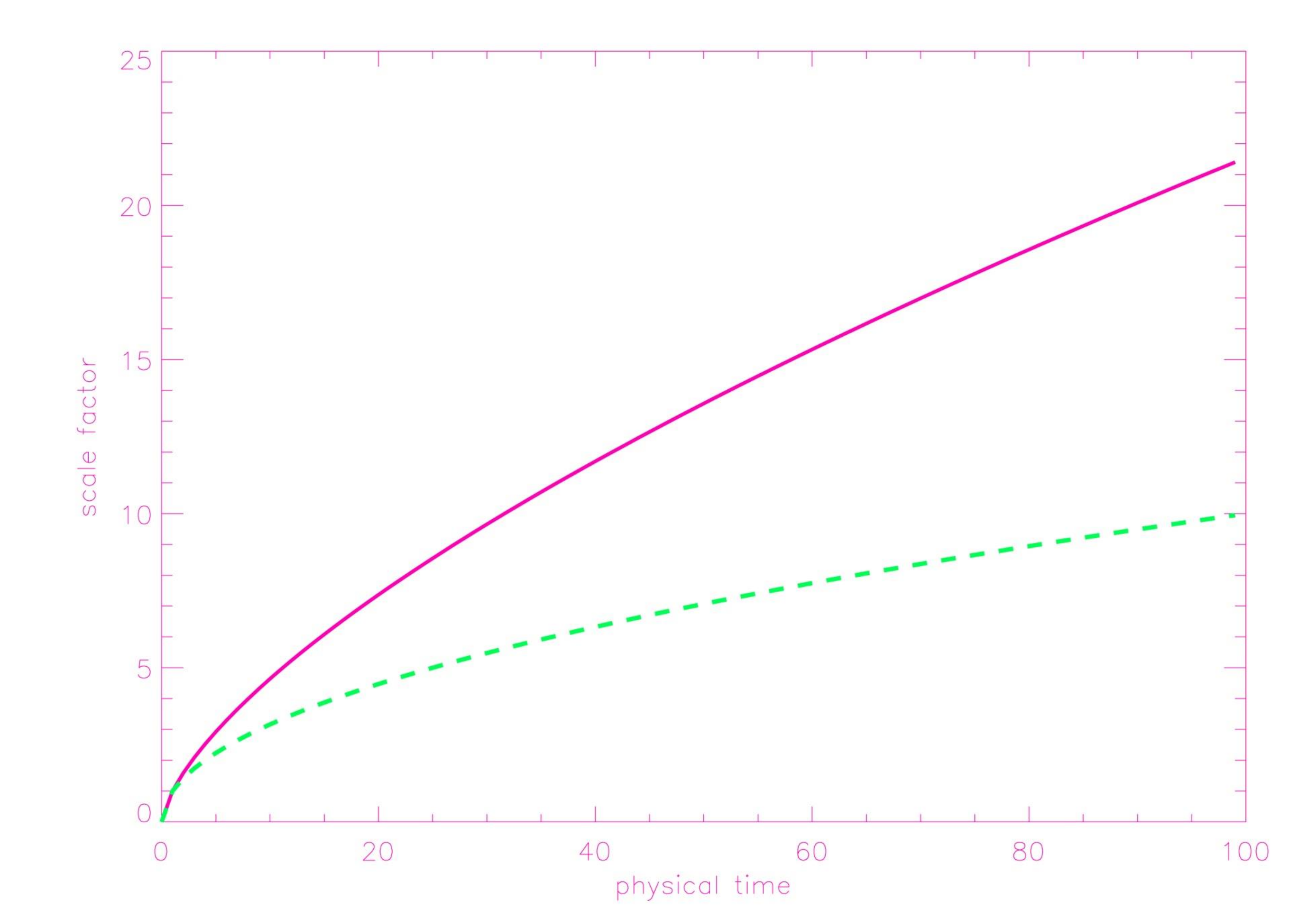
If k=0 the equation becomes

$$\frac{1}{2}\left(\frac{da}{dt}\right)^2 = \frac{GM\ a_0}{a^2}$$

where $$M = \tfrac{4\pi}{3}\rho_0 a_0^3$$

and solution is

$$a(t) = a_0\sqrt{t/t_0}$$



25
20
15
10
5
0
0
20
40
60
80
100
scale factor
physical time

The vacuum dominated Universe

$p = -\rho c^2$ and $k = 0$

in this case Friedmannien equations are :

the first is :

$$\frac{1}{2}\dot{a}^2 = \frac{4\pi}{3} G a^2 \rho$$

and the second is :

$$\ddot{a} = \frac{8\pi G}{3}\rho a$$

results repulsion instead of attraction!

the density of this media is constant

$$\frac{d\rho}{dt} = 0 \quad \Longrightarrow \rho = const$$

Let introduce definition

$$H^2 \equiv \frac{8\pi G}{3}\rho$$

and call it the Hubble parameter

In this case the solution of Friedmannien equations is

$$a(t) = a_0 e^{Ht}$$

The Standard Cosmological Model

- The Standard Cosmological Model is:
- the model of expanding Universe with flat hypersurface which is filled by different types of matter: small amount of relativistic matter (photons), baryonic matter and dark matter which also obey dust like equation of state, and dark energy or quintessense which obeys vacuum dominated equation of state.

WMAP

Old Universe – *New* Numbers

$\Omega_{tot} = 1.02^{+0.02}_{-0.02}$
$w < -0.78$ (95% CL)
$\Omega_{\Lambda} = 0.73^{+0.04}_{-0.04}$
$\Omega_b h^2 = 0.0224^{+0.0009}_{-0.0009}$
$\Omega_b = 0.044^{+0.004}_{-0.004}$
$n_b = 2.5 \times 10^{-7}{}^{+0.1\times10^{-7}}_{-0.1\times10^{-7}}$ cm^{-3}
$\Omega_m h^2 = 0.135^{+0.008}_{-0.009}$
$\Omega_m = 0.27^{+0.04}_{-0.04}$
$\Omega_{\nu} h^2 < 0.0076$ (95%CL)
$m_{\nu} < 0.23$ eV (95% CL)
$T_{cmb} = 2.725^{+0.002}_{-0.002}$ K
$n_{\gamma} = 410.4^{+0.9}_{-0.9}$ cm^{-3}
$\eta = 6.1 \times 10^{-10}{}^{+0.3\times10^{-10}}_{-0.2\times10^{-10}}$
$\Omega_b \Omega_m^{-1} = 0.17^{+0.01}_{-0.01}$
$\sigma_8 = 0.84^{+0.04}_{-0.04}$ Mpc
$\sigma_8 \Omega_m^{0.5} = 0.44^{+0.04}_{-0.05}$
$A = 0.833^{+0.086}_{-0.083}$

$n_s = 0.93^{+0.03}_{-0.03}$
$dn_s/d \ln k = -0.031^{+0.016}_{-0.018}$
$r < 0.71$ (95% CL)
$z_{dec} = 1089^{+1}_{-1}$
$\Delta z_{dec} = 195^{+2}_{-2}$
$h = 0.71^{+0.04}_{-0.03}$
$t_0 = 13.7^{+0.2}_{-0.2}$ Gyr
$t_{dec} = 379^{+8}_{-7}$ kyr
$t_r = 180^{+220}_{-80}$ Myr (95% CL)
$\Delta t_{dec} = 118^{+3}_{-2}$ kyr
$z_{eq} = 3233^{+194}_{-210}$
$\tau = 0.17^{+0.04}_{-0.04}$
$z_r = 20^{+10}_{-9}$ (95% CL)
$\theta_A = 0.^\circ598^{+0.002}_{-0.002}$
$d_A = 14.0^{+0.2}_{-0.3}$ Gpc
$l_A = 301^{+1}_{-1}$
$r_s = 147^{+2}_{-2}$ Mpc

$$\rho = \rho_m + \rho_\Lambda + \rho_r$$

$$p = p_m + p_\Lambda + p_r$$

the evolution of the different

media components runs independently

and $p_m = 0,$

$$p_\Lambda = -\rho_\Lambda c^2$$

$$p_r = \frac{1}{3}\rho_r c^2$$

and three Friedmannien equations are:

$$\frac{1}{2}\dot{a}^2 = \frac{4\pi}{3}Ga^2(\rho_m + \rho_\Lambda + \rho_r)$$

$$\ddot{a} = -\frac{4\pi G}{3}\left(\rho_m - 2\rho_\Lambda + 2\rho_r\right)a$$

$$\frac{d(\rho_m + \rho_r)}{dt} = -3H(\rho_m + \frac{4}{3}\rho_r)$$

One can put densities as functions of redshift and rewrite these equations as

$$\rho_m = \rho_{0m}\left(\frac{a_0}{a(t)}\right)^3 \quad \text{and} \quad \rho_m = \rho_{0m}(1+z)^3$$

$$\rho_r = \rho_{0r}\left(\frac{a_0}{a(t)}\right)^3 \quad \text{and} \quad \rho_r = \rho_{0r}(1+z)^3$$

$$\rho_\Lambda = const$$

According to WMAP the total density of our Universe is:

$$\Omega_{total}=1$$

and contribution of different type of matter in density is:

$\Omega_{0m}=0.27$ and $\Omega_q=0.73$

Therefore, the first Friedmannien equation is:

$$\frac{\dot{a}^2}{a^2} = H_0^2\left[\Omega_{0m}(1+z)^3 + \Omega_q + \Omega_{0r}(1+z)^4\right]$$

here Ω_{0r} is present density of the CMBR with respect to critical density and the second Friedmannien equation becomes

$$\ddot{a} = -\frac{1}{2} H_0^2 \left(\Omega_{0m} (1+z)^3 - 2\Omega_\Lambda + 2\Omega_{0r} (1+z)^4 \right) a$$

and we have two regimes

$$1 + z > \left(\frac{2\Omega_\Lambda}{\Omega_{0m}} \right)^{1/3} \quad \text{is desceleration stage}$$

$$1 + z < \left(\frac{2\Omega_\Lambda}{\Omega_{0m}} \right)^{1/3} \quad \text{is acceleration stage}$$

$z \approx 0.7$

RED SHIFT

- The red shift is most known cosmological phenomena.

$$\frac{\Delta\nu}{\nu} = \frac{\Delta\lambda}{\lambda} = \frac{v}{c} = z$$

v is velocity of an emitter which is moving from observer

$$\Delta\nu = \nu_e - \nu_o$$

$$\Delta\lambda = \lambda_o - \lambda_e$$

The general description of the phenomena is as follows: a spectral line from another galaxy ia emitted with the same frequency as in laboratory. But observed frequency is different.

Let consider the motion of light rays in the expanding Universe. The light ray is moving along a straight line according to the equation ds=0. This equation is postulate of the Special Relativity Which is valid in general Relativity too. In the expanding Universe metric has form (flat hypersurface):

$$ds^2 = c^2 dt^2 - a^2(t)\left(dr^2 + r^2\left\{d\theta^2 + \sin^2\theta d\varphi^2\right\}\right)$$

Let assume that an observer is in the center of spherical coordinate system and the light rays move along the radial coordinate. So, we can put dθ=0, dϕ=0.

In this case metric equation is reduced to form

$$dt^2 - a^2(t)r^2 = 0$$ or

$$dt = \pm a(t)r$$

One can solve this equation as

$$\int_{t_e}^{t_o} \frac{dt}{a(t)} = -r_o + r_e$$

In this equation *t* is physical time. One can introduce new variable η which is called conformal time. The equation for this time is $d\eta = dt/a(t)$. In this case the solution of above equation is very simple

$$\eta_e - \eta_o = r_e - r_o$$

Now one can calculate the interval of an event in emitter and in observer. Suppose that the lagrangian distance (r) between the emitter and the observer is constant. In this case interval of an event at emitter position ($\Delta\eta$) is equal to interval at observer position:

$$\Delta\eta_e = \Delta\eta_o$$

One can rewrite this equation in terms of physical time as

$$\frac{dt_e}{a(t_e)} = \frac{dt_o}{a(t_o)}$$

Suppose that the event is one cycle of radiation. One can rewrite above equation in terms of frequency

$$\nu_o = \frac{a(t_e)}{a(t_o)} \nu_e$$ or

$$\frac{a(t_o)}{a(t_e)} = 1 + z$$

as general definition of red shift

$$\eta_e - \eta_o = \frac{3}{a_o}\left[t_o - \sqrt[3]{t_e t_o^2}\right]$$

$$l_e = a_o r_e$$

$$H = \frac{1}{a(t)}\frac{da(t)}{dt} = \frac{2}{3}\frac{1}{t}$$

$$1 + z = \left(\frac{t_o}{t_e}\right)^{2/3}$$

$$\frac{Hl_e}{2c} = 1 - \frac{1}{\sqrt{1+z}}$$

or

$$z = \frac{Hl_e}{2c}\frac{\left(2 - \frac{Hl_e}{2c}\right)}{\left(1 - \frac{Hl_e}{2c}\right)^2}$$

for

$$\frac{Hl_e}{2c} << 1$$

one obtain

$$z = \frac{Hl_e}{c}$$

what is Hubble law, in the case $z >> 1$

one obtain

$$\frac{Hl_e}{2c} \to 1$$

The definition of cosmic distance is:

$$r_e = a_o \int_{t_e}^{t_o} \frac{dt}{a(t)}$$

One can rewrite the definition for red shift in form:

$$\frac{a_o}{a(t)} = 1 + z(t)$$

and rewrite the cosmic distance in terms of red shift

$$r_e = -\int_0^z (1 + z)dt(z)$$

Also one can rewrite the equation for red shift in the form

$$a(t) = \frac{a_o}{1 + z(t)}$$

and obtain the Hubble parameter as function of red shift

$$H(t)dt = -\frac{dz}{1 + z}$$

and one can substitute these equations into definition of cosmic distance and obtain the equation which determines the distance to object as a function of its red shift:

$$r_e = \frac{c}{H}\int_0^z \frac{dz}{\sqrt{\Omega_{mo}(1+z)^3 + \Omega_q + \Omega_{ro}(1+z)^4}}$$

The End