Five-color theorem, black hole mass and pre-holography

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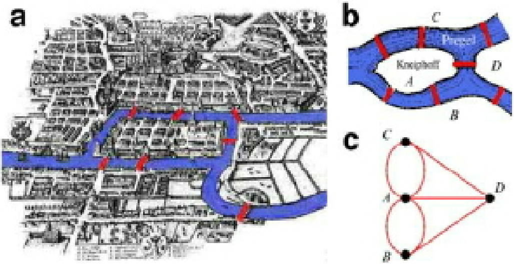
MSU, Nov 27, 2018

Topology

- **Seven bridges of Königsberg (1736)**



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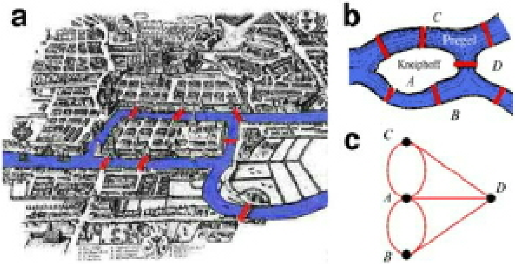

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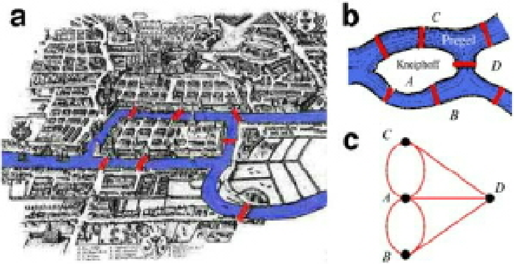

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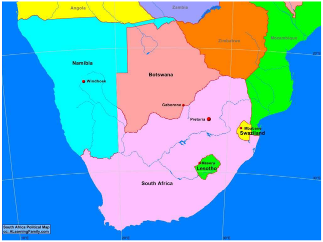
- **Proof given by Heawood in 1890.**

Five-color theorem

- **Countries are simply-connected regions**



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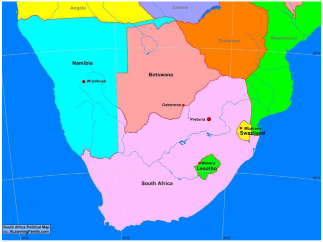

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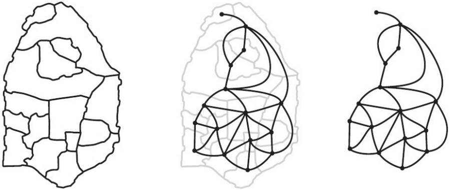

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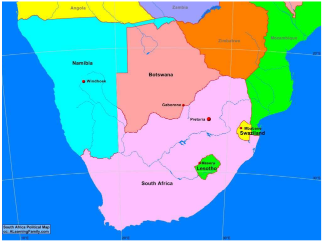

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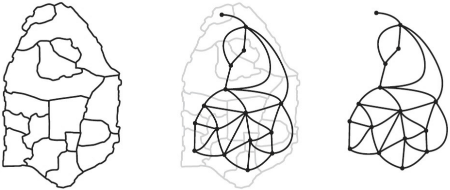

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- **Total Action**

$$\tilde{I} = -\frac{1}{4}\int\limits_M dtd^3x\left(F^{\mu\nu}F_{\mu\nu} + \gamma\ {}^*F^{\mu\nu}F_{\mu\nu}\right).$$

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- **Alternative: asymptotic (anti) self-duality in $F^{\mu\nu}$**

$$F^{\mu\nu} = \pm \, {}^\ast F^{\mu\nu} \quad \text{at} \ \ \partial M$$

fixes coupling as $\gamma = \mp 1$

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- **Arbitrary Quadratic Curvature Couplings**

$$I = \frac{1}{16\pi G} \int\limits_M d^4x \sqrt{-g} \left(R - 2\Lambda + \alpha R^{\mu\nu} R_{\mu\nu} + \beta R^2 \right)$$

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- **Gauss-Bonnet term (topological invariant in 4D)**

$$GB = \sqrt{-g} \left(Rie^2 - 4R^{\mu\nu} R_{\mu\nu} + R^2 \right)$$

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$$E^{\mu\nu}_{(\beta)} = 2R\left(R^{\mu\nu} - \frac{1}{4} g^{\mu\nu} R\right) + 2\left(g^{\mu\nu}\Box - \nabla^{\mu}\nabla^{\nu}\right) R.$$

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$$R^{\mu\nu}_{\alpha\beta} = -\frac{1}{\ell^2_{\rm eff}}\,\delta^{\mu\nu}_{\alpha\beta}$$

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$$n_{\mu}\Theta^{\mu}_{(\alpha)}(\delta g,\delta\Gamma)=\frac{\alpha}{16\pi G}\delta^{[\mu\nu\delta]}_{[\sigma\lambda\gamma]}\left[-n_{\mu}G^{\gamma}_{\delta}g^{\lambda\varepsilon}\delta\Gamma^{\sigma}_{\nu\varepsilon}+n^{\lambda}\nabla_{\mu}G^{\gamma}_{\delta}\left(g^{-1}\delta g\right)^{\sigma}_{\nu}\right]$$

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- β part

$$n_\mu \Theta^\mu_{(\beta)}(\delta g, \delta\Gamma) = \frac{\beta}{8\pi G}\delta^{[\mu\nu]}_{[\sigma\lambda]}\left[n_\mu R g^{\lambda\varepsilon}\delta\Gamma^\sigma_{\nu\varepsilon} - n^\lambda \nabla_\mu R \left(g^{-1}\delta g\right)^\sigma_\nu\right]$$

Abbott-Deser-Tekin energy

- **Perturbation around a background metric** $\bar{g}_{\mu\nu}$

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

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- **Linearized curvatures**

$$R^L_{\mu\nu} = \frac{1}{2}\left(\bar{\nabla}_\mu \bar{\nabla}_\nu - \bar{\Box} h_{\mu\nu}\right)$$

$$R^L = \Lambda h$$

Abbott-Deser-Tekin energy

- **Conserved quantities**

$$
\begin{aligned}
8\pi G\, Q^{\mu}_{ADT}[\bar{\xi}] \;=\;& [1+2\Lambda(\alpha+4\beta)] \int\limits_{\partial M} d^3x\, G_L^{\mu\lambda}\bar{\xi}_\lambda + \\
&+(\alpha+2\beta)\int\limits_{\Sigma} dS_\nu \left(2\bar{\xi}^{[\mu}\bar{\nabla}^{\nu]}R^L + R^L\bar{\nabla}^{\mu}\bar{\xi}^{\nu}\right) - \\
&-\alpha\int\limits_{\Sigma} dS_\nu \left(2\bar{\xi}_\lambda\bar{\nabla}^{[\mu}G_L^{\nu]\lambda} + 2G_L^{\lambda[\mu}\bar{\nabla}^{\nu]}\bar{\xi}_\lambda\right) .
\end{aligned}
$$

Abbott-Deser-Tekin energy and Criticality

- Schwarzschild-AdS black hole

$$M = m\left(1 + 2\Lambda(\alpha + 4\beta)\right).$$

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  `[H. Lu and C.N. Pope, arXiv:1101.1971]`

$$I_{critical} = \frac{1}{16\pi G}\int\limits_{M} d^4x\sqrt{-g}\left[R - 2\Lambda + \frac{3}{2\Lambda}\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right)\right]$$

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  -conserved charges coming from linearization of the theory
  -evaluation on particular black hole solutions

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- Zero black hole energy:
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  -evaluation on particular black hole solutions
- Linearization instability
  [E. Altas and B. Tekin, arXiv:1705.10234]

Noether-Wald charges

- **For a gravity Lagrangian** $\mathcal{L} = \mathcal{L}(g_{\mu\nu}, R_{\mu\nu\alpha\beta})$, the Noether current is

$$J^{\alpha}[\xi] = 2E^{\alpha\beta}_{\mu\nu}(g^{\nu\lambda}\delta_{\xi}\Gamma^{\mu}_{\beta\lambda}) + 2\nabla^{\mu}E^{\alpha\beta}_{\mu\nu}(g^{\nu\lambda}\delta_{\xi}g_{\lambda\beta}) + \mathcal{L}\xi^{\mu} \,,$$

  where

$$E^{\mu\nu\alpha\beta} = \frac{\delta L}{\delta R_{\mu\nu\alpha\beta}} \,.$$

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- **Direct application of Wald formula to EH+QCG fails** 😭

Noether-Wald charge

- **Noether-Wald current**

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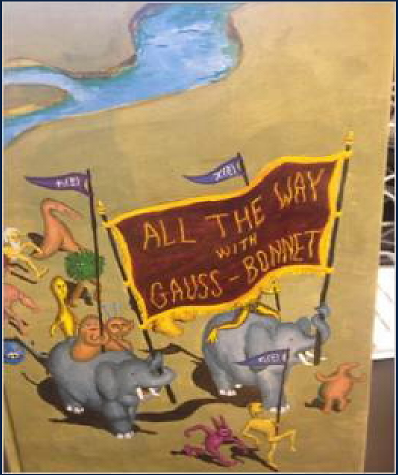
All the way with Gauss-Bonnet (Spivak)



EH+QCG+GB

- **Addition of GB with a coupling** $\gamma/16\pi G$

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EH+QCG+GB

- **Noether-Wald charges for gravity+GB**

$$q_{top}^{\alpha\beta} = q_{NW}^{\alpha\beta} + \frac{\ell^2}{8}\left[1 + 2\Lambda\left(\alpha + 4\beta\right)\right]\delta_{[\mu\nu\sigma\lambda]}^{[\alpha\beta\gamma\delta]}\nabla^{\mu}\xi^{\nu}R_{\gamma\delta}^{\sigma\lambda}$$

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- **...with no tears!!!** 😃

Non-Einstein spaces

- **Gravitational waves** [Podolsky, gr-qc/9801052]

$$ds^2 = \frac{\ell^2}{z^2}\left[-\left(1+F(t,z,x)\right)dt^2 + 2dtdu + dz^2 + dx^2\right]$$

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- **Those boundary terms can be used to obtain a holographic description of the theory.**

Topological Invariants and AdS/CFT

- **Einstein+Gauss-Bonnet in 4D ($\Lambda = -3/\ell^2$)**

$$I = \frac{1}{16\pi G} \int_M d^4x \sqrt{-g} \left[R - 2\Lambda + \gamma \left(R^{\alpha\beta\mu\nu} R_{\alpha\beta\mu\nu} - 4 R^{\mu\nu} R_{\mu\nu} + R^2 \right) \right]$$

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- **Euclidean action for Sch-AdS black hole:**

$$G = \beta^{-1} I^{Eucl} = \frac{M}{2} \left(1 + \frac{4}{\ell^2} \gamma \right) - TS + \frac{\pi r^3}{4G\ell^2} \left(1 - \frac{4}{\ell^2} \gamma \right)$$

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- **Extrinsic counterterms (Kounterterms)**
  `R.O., [hep-th/0504233,hep-th/0610230]`

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Variational problem in GB gravity

- **For any $D > 4$ (arbitrary GB coupling γ)**

$$\delta I_{GB} = \frac{\gamma}{4\pi G} \int\limits_{\partial M} d^{D-1}x \sqrt{-h}\, \delta^{[jj_1j_2]}_{[ii_1i_2]} \left[\frac{1}{2} \left(h^{-1}\delta h \right)^i_k K^k_j + \delta K^i_j \right] \left(\frac{1}{2} \mathcal{R}^{i_1i_2}_{j_1j_2}(h) - K^{i_1}_{j_1} K^{i_2}_{j_2} \right)$$

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- **No Gibbons-Hawking term for $4D$ GB $\Rightarrow$ No quasilocal stress tensor**

$$\delta I=\int\limits_{\partial M} d^3x\sqrt{-h}\left(\frac{1}{2}\tau^j_i\left(h^{-1}\delta h\right)^i_j+\Delta^j_i\,\delta K^i_j\right)$$

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- Expansion of K_i^j for any AAdS spacetime

$$\begin{aligned} K^i_j &= \frac{1}{\ell}\delta^i_j - \ell S^i_j(h) + \mathcal{O}(\mathcal{R}^2) \\ S^i_j(h) &= \frac{1}{D-3}(\mathcal{R}^i_j(h) - \frac{1}{2(D-2)}\delta^i_j\mathcal{R}(h)) \end{aligned}$$

From Extrinsic to Intrinsic Counterterms

- `O. Miskovic and R.O., [arXiv:0902.2082]`

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- **Balasubramanian-Kraus counterterms in 4D**

$$\mathcal{L}_{ct} = \frac{1}{8\pi G}\sqrt{-h}\left(\frac{2}{\ell} + \frac{\ell}{2}\mathcal{R}(h)\right) + \ldots$$

Renormalized Einstein-AdS action

- **MacDowell-Mansouri (Stelle-West) form of the action**

$$I_{ren} = \frac{\ell^2}{256\pi G} \int\limits_M d^4x \sqrt{-g}\, \delta^{[\sigma\lambda\mu\nu]}_{[\gamma\delta\alpha\beta]} \left(R^{\gamma\delta}_{\sigma\lambda} + \frac{1}{\ell^2} \delta^{[\gamma\delta]}_{[\sigma\lambda]} \right) \left(R^{\alpha\beta}_{\mu\nu} + \frac{1}{\ell^2} \delta^{[\alpha\beta]}_{[\mu\nu]} \right).$$

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`G. Anastasiou and R.O., [arXiv:1608.07826]`

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$$g_{(2)ij} = -\frac{1}{d-2}\left(\mathcal{R}_{(0)ij} - \frac{1}{2(d-1)}\mathcal{R}g_{(0)ij}\right)$$

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